MATHEMATICS SOLUTION
(CBCGS SEM – 4 DEC 2018)
BRANCH – EXTC ENGINEERING

1 a) If $A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$, then find the eigen values of of 6 $A^{-1} + A^3 + 2I$. **(05)**

Ans. $A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$,

Since A is an upper triangular matrix, eigen values (λ) =diagonal elements =2,3
We know, $f(\lambda)$ is eigen value of $f(A)$

Let $f(A) = 6A^{-1} + A^{-3} + 2I$
$\therefore f(\lambda) = 6\lambda^{-1} + \lambda^3 + 2$
$= \frac{6}{\lambda} + \lambda^3 + 2$
When $\lambda = 2$, $f(2) = \frac{6}{2} + 2^3 + 2 = 13$
When $\lambda = 3$, $f(3) = \frac{6}{3} + 3^3 + 2 = 31$

$\therefore$ Eigen values of 6 $A^{-1} + A^3 + 2I$ are 13, 31.

1 b) Find a vector orthogonal to both u = (-6,4, 2) and v (3,1,5). **(05)**

Ans. u = [- 6,4,2],v = [3,1,5]
Let $w = (w_1\ w_2 w_3)$ be a vector which is orthogonal to both u and v.Type equation here.
$\therefore \langle u \quad w \rangle = 0 \Rightarrow -6w_1 + 4w_2 + 2w_3 = 0$ and,
$\therefore \langle v \quad w \rangle = 0 \Rightarrow -3w_1 + 1w_2 + 5w_3 = 0$

By Crammer's Rule, $\frac{w_1}{\begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix}} = \frac{-w_2}{\begin{vmatrix} -6 & 2 \\ 3 & 5 \end{vmatrix}} = \frac{w_3}{\begin{vmatrix} -6 & 4 \\ 3 & 1 \end{vmatrix}}$

$\therefore \frac{w_1}{18} = \frac{-w_2}{-36} = \frac{w_3}{-18}$

$\therefore \frac{w_1}{1} = \frac{w_2}{1} = \frac{w_3}{-1}$

Let, $\frac{w_1}{1} = \frac{w_2}{1} = \frac{w_3}{-1} = t$

$\therefore w_1 = t;\ \ w_2 = 2t;\ \ w_3 = -t$

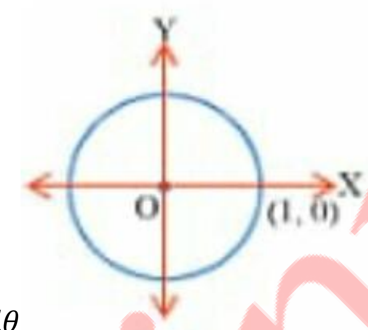
$\therefore w = [t\ 2t\ -t]$
∴ For t = 1, = w = [1 2 -1]

Hence, a vector orthogonal to both u and v is w = [1 2 -1]

1 c) Evaluate $\int_c \log zdz = 2\pi i$, where C is the until circle in the z plane. **(05)**

Ans. The unit circle in the z plane is |z| = 1 with Centre (0,0) and radius =1
Let $I = \int_c \log zdz$

Put $z = re^{i\theta} = 1e^{i\theta} = e^{i\theta}$



$\therefore dz = e^{i\theta}.id\theta$

$\therefore I = \int_0^{2\pi} log(e^{i\theta}).e^{i\theta}.\ id\theta$

$= \int_0^{2\pi} i\ \theta \log e.\ e^{i\theta}\ id\theta$

$= i^2 \int_0^{2\pi} \theta\ e^{i\theta}\ d\theta$

$= i^2 \left[\theta \frac{e^{i\theta}}{i} - 1.\frac{e^{i\theta}}{i^2}\right]_0^{2\pi}$

$= \left[i\theta\ e^{i\theta} - e^{i\theta}\right]_0^{2\pi}$

$=\left[e^{i\theta}(i\theta-1)\right]_0^{2\pi}$

$= e^{i2\pi}(i.2\pi-1)-e^{0}(i.0-1)$

= (cos 2π + i sin 2 π)(2π i -1) +1

= (1+0) (2 π i -1) +1

= 2 π i -1 + 1

Hence, $\int_c \log z\, dz = 2\pi i$

1 d) Let x be continuous random variable with distribution **(05)**

$f(x)=\begin{cases} x/6+k & 0\le x\le 0 \\ 0 & elsewhere \end{cases},$

Evaluate k and find P (1 $\le x \le 2$) and P (1 $\le x \le 3$)

Ans: For any probability density function $,\int_{-\infty}^{\infty} f(x)\,dx = 0$

$$\therefore \int_0^3 \left(\frac{x}{6}+k\right) dx = 1$$

$\therefore \left[\frac{1}{6}.\frac{x^2}{2}\ kx\right]_0^3 = 1$

$\therefore \left(\frac{3^2}{12}+3k\right) - 0 = 1$

$\therefore \frac{3}{4} + 3k = 1$

$\therefore k = \frac{1}{12}$

$\therefore f(x) = \frac{x}{6}+\frac{1}{12}$

$\therefore$ P $(1 \le X \le 2) = \int_1^2 \left(\frac{x}{6}+\frac{1}{12}\right) dx$

= $\left[\frac{1}{6} \cdot \frac{x^2}{2} + \frac{1}{12} \ \ . x\right]_1^2$

= $\left[\frac{2^2}{12} + \frac{2}{12}\right]$ - $\left[\frac{1^2}{12} + \frac{1}{12}\right]$

= $\frac{1}{3}$

And, P(1$\leq X \leq 3$) = $\int_1^3 \left(\frac{x}{6} + \frac{1}{12}\right)\ dx$

= $\left[\frac{1}{6} \cdot \frac{x^2}{2} + \ \frac{1}{12} \cdot x\ \right]_1^3$

= $\left[\frac{3^2}{12} + \frac{3}{12}\right]$ - $\left[\frac{1^2}{12} + \frac{1}{12}\right]$

= $\frac{5}{6}$

Hence, k = $\frac{1}{12}$; P (1 $\leq X \ \leq 2$) ; P (1 $\leq X \leq 3$) = $\frac{5}{6}$

2 a) Show that A is diagonalizable .Also find the transforming matrix M and the diagonal matrix D where A = $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ (08)

Ans: Let λ be eigen value and X be corresponding eigen vector of matrix A.

Characteristic equation is |A – λ I | = 0

$$\therefore \begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix}$$

On solving we get

λ^3(sum of diagonal elements) λ^2 + (sum of the minors of diagonal elements) $\lambda - |A| = 0$

$\therefore\ \lambda^3$ - (-9 + 3 + 7) λ^2 + (-11+ 1 + 5) $\lambda - 3 = 0$

$\therefore \lambda^3 - \lambda^2 - 5\lambda - 3 = 0$

$\therefore$ Eigen values (λ) are -1 ,-1,3

Case 1: $\lambda = -1$

$\therefore [A - \lambda I]$ X = 0

$$\therefore \begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 - R_1 ; R_3 - 2R_1 ; \frac{1}{4}R_1 \Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

Numbers of unknowns (n) = 3

Rank (r) = number of non – zero rows =1

Algebraic multiplicity (A.M)

= No. of times "λ = - 1" is repeated = 2

Geometric multiplicity (G.M) = n-r = 3 -1 =2

$\therefore$A . M = G.M for " λ= - 1 "

Expanding (1), $-2x_1 + x_2 + x_3 = 0$

Put $x_1 = t$ and $x_2 = s$

$\therefore -2t + s + x_3 = 0$

$\therefore x_3 = 2t - s$

$$\therefore \text{Eigen vector} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ s \\ 2t - s \end{bmatrix} = \begin{bmatrix} 1t + 0s \\ 0t + 1s \\ 2t - 1s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} s$$

$\therefore$ Eigen vector X_1 = [1 0 2]'&X_2 = [0 1 -1]'

Case 2; λ = 3

$\therefore$[A – λ I] X = 0

$$\therefore \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \quad - \frac{2}{3} R_1; \; R_3 - \frac{4}{3} R_1; \; \frac{1}{4} R_1$$

$$\Rightarrow \begin{bmatrix} -3 & 1 & 1 \\ 0 & -8/3 & 4/3 \\ 0 & 8/3 & -4/3 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 + R_2; \; \frac{3}{4} R_2 \; \Rightarrow \begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (3)$$

Here n = 3 and r = 1
A.M = No. of times "$\lambda = 3$" is repeated =1
G.M = n-r = 3 – 2 = 1
∴ A.M = G.M .for "$\lambda = 3$ "

Expanding (3),

-3$x_1 + x_2 + x_3 = 0 \;\rightarrow (4)$&
-2$x_2 + x_3 = 0 \rightarrow (5)$

From (5), $x_3 = 2x_2$
From (4), - 3$x_1 + x_2 + 2x_2 = 0$
∴ 3 x_2= 3x_1

$$\therefore x_1 = x_2$$

$$\therefore \text{Eigen vector} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} x_2$$

∴ Eigen vector $X_2 = \;[I \;\; 1 \; 2\,]'$

Since A.M = G.M for all eigen values, matrix A is diagonalizable .
∴ M^{-1} A$M = D$

So the given matrix A is diagonalized to Diagonal Matrix. D by the Transforming Matrix M. where

$\boldsymbol{D}$ = $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ **and** $\boldsymbol{M}$ = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

2 b)Find the extremal of the function $\int_0^{\pi/2} (y'^2 - y^2 + 2yx)dx$ with y (0) = 0 &(06)

$y\left(\frac{\pi}{2}\right)$ **= 0.**

Ans: Let $\int_{x_1}^{x_1} F dx = \int_0^{\pi/2} (y'^2 - y^2 + 2yx)dx$

$\therefore F = y'^2 - y^2 + 2yx$

$$\therefore \frac{\partial F}{\partial y} = -2y + 2x - 0\;; \quad \& \frac{\partial F}{\partial y'} \;\; 2y'\;;$$

By Euler's Lagrange equation ,the necessary condition for the given functional to be extremum is $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$\therefore 2\,x - 2\,y - \frac{d}{dx}(2y') = 0$

$\therefore x - y - \frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$ (Dividing by 2)

$$\therefore \frac{d^2y}{dx^2} + y = x$$

$$\therefore D^2y + y = x \left(where, D = \frac{d}{dx}\right)$$
$$\therefore \quad (D^2 + 1)\, y = x$$

This is Liner Differential Equation

Auxiliary equation is $D^2 + 1 = 0$
$\therefore D = \pm i = 0 \pm 1i$

$\therefore$ Complimentary Function is
$y_{\circ} = e^{0x}\, (c_1 \cos 1\, x + c_2 \sin\ )$

Particular Integral is $y_p = \frac{1}{f(D)} X$Where , X = x

$\therefore y_p = \frac{1}{(D^2+1)} (x)$

= $(1 + D^2)^{-1} (x)$

= (1 - $D^2 + D^4 - \cdots$.) x

= x - $D^2 x + D^4 x - \cdots$.

= $x + 0 + 0 + \cdots$

= x

$\therefore$ The Complete Solution is $y = y_c + y_p$

Hence the solution is $y = c_1 \cos x + c_2 \sin x + x \rightarrow (1)$

Given, y (0) = 0

$\therefore$ Put x = 0 and y = 0 in (1)

$\therefore$ 0 = $_1$. cos0 + c_2 .sin0 + 0

$\therefore$ 0 = c_1 . 1 + c_2 .0

$\therefore$ 0 = c_1

Given, $y\left(\frac{\pi}{2}\right)$ = 0

$\therefore$ Put x= $\frac{\pi}{2}$, c_1 = 0 and y = 0 in (1)

$\therefore$ 0 = 0 + $c_2$sin $\frac{\pi}{2} + \frac{\pi}{2}$

$\therefore$ 0 = c_2. 1 + $\frac{\pi}{2}$

$\therefore c_2$ = -$\frac{\pi}{2} \rightarrow (3)$

$\therefore$ From (1), (2) and (3), $y = 0 - \frac{\pi}{2}\sin x + x$

$\therefore$ The extremal of the given function is

$\boldsymbol{y = x - \frac{\pi}{2}\sin x}$

2 c) Let R^4 have the Euclidean inner product .use Gram -Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ in to an orthonormal basis where u_1 = (1,0,1,1); u_2 = (-1,0,-1,1); u_3 = (0,-1,1,1).
(08)

Ans:<u>**Gram Schmidt orthogonalization:**</u>

u_1 = (1,0,1,1); u_2 = (-1,0,-1,1); u_3 = (0,-1,1,1):

<u>**S1:**</u>
Let $v_1 = u_1$ = (1,0,1,1)

$\therefore \|v_1\|^2 = (1)^2 + (0)^2 + (1)^2 + (1)^2 = 3$ and
$\langle u_2, v_1 \rangle$ = (-1) (1) + (0) (0) + (1) (-1) + (1)(1) = -1

<u>**S2:**</u>
Let $v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

= (-1,0,-1,1)- $\frac{(-1)}{3} \times$(1,0,1,1,)

= (-1,0,-1,1) + $\left(\frac{1}{3}, 0\ \frac{1}{3}, \frac{1}{3}\right)$

= $\left(\frac{-2}{3}, 0\ \frac{-2}{3}, \frac{4}{3}\right)$

$\therefore \|v_2\|^2 = \left(\frac{-2}{3}\right)^2 + (0)^2 + \left(\frac{-2}{3}\right)^2 + \left(\frac{4}{3}\right)^2 = \frac{8}{3}$;

Now ,

$\langle u_{3,} v_1 \rangle$ = (0)(1) + (-1)(0) + (1)(1) + (1)(1) =2 and

$\langle u_{3,} v_2 \rangle$ = (0)$\left(\frac{-2}{3}\right) + (-1)(0) + (1)\left(\frac{-2}{3}\right) + (1)\left(\frac{4}{3}\right) = \frac{2}{3}$

S3:

Let $v_3 = u_{3,} - \frac{\langle u_{3,} v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_{3,} v_2 \rangle}{\|v_2\|^2} v_2$

$= (0,-1,1,1) - \frac{2}{3}(1,0,1,1) - \frac{2/3}{8_{4}/3}\left(\frac{-2}{3}, 0, \frac{-2}{3}, \frac{4}{3}\right)$

∴ **Orthonormal Bases are:**

$\underset{q_1}{\rightarrow} = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \frac{1}{\sqrt{3}}(1,0,1,1) = \left(\frac{\sqrt{3}}{3}, 0, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$

$\underset{q_2}{\rightarrow} = \frac{\vec{v_2}}{\|\vec{v_2}\|} = \frac{1}{\sqrt{8/3}}\left(\frac{-2}{3}, 0, \frac{-2}{3}, \frac{4}{3}\right)$

$= \frac{\sqrt{3}}{\sqrt{8}}\left(\frac{-2}{3}, 0, \frac{-2}{3}, \frac{4}{3}\right) = \left(\frac{-\sqrt{6}}{6}, 0, \frac{-\sqrt{6}}{6}, \frac{\sqrt{6}}{3}\right)$

And, $\underset{q_3}{\rightarrow} = \frac{\vec{v_3}}{\|\vec{v_3}\|} = \frac{1}{\sqrt{3/2}}\left(\frac{-1}{2}, -1, \frac{1}{2}, 0\right)$

$= \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{-1}{2}, -1, \frac{1}{2}, 0\right) = \left(\frac{-\sqrt{6}}{6}, \frac{-\sqrt{6}}{3}, \frac{\sqrt{6}}{6}, 0\right)$

∴ **Orthonormal basis of the subspace S are**

$$\left\{\left(\frac{\sqrt{3}}{3}, 0, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right); \left(\frac{-\sqrt{6}}{6}, 0, \frac{-\sqrt{6}}{6}, \frac{-\sqrt{6}}{3}\right); \left(\frac{-\sqrt{6}}{6}, \frac{-\sqrt{6}}{3}, \frac{\sqrt{6}}{6}, 0\right)\right\}$$

3 a) The number of accidents in a year attributed to taxi drivers in a city follows Poisson distribution with mean 3 out of 1000 taxi drivers, find approximately the number of drivers with (06)

1) **No accidents in a year**
2) **More than 3 accidents in a year.**
   (Given: $e^{-1} = 0.3679, e^{-2} = 0.1353, e^{-3} = 0.0498$)

Ans: Let X denote the number of accidents in a year due to taxi drivers .

Given X follows Poisson distribution .

Average number of accidents in a year due to taxi drivers =m=3

N = 1000

$$\therefore\ P\ (X = x) = \frac{e^{-m} m^x}{x!} = \frac{e^{-3} 3^x}{x!}$$

i) Probability a driver has no accidents in a year = P (X = 0)

$= \frac{e^{-3} 3^0}{0!}$

= 0.04979

No. of drivers with no accidents in a year = N × P (X = 0)

= 1000 × 0.0.4979
= 49.8
= 50 drivers

ii) Probability a driver has more than 3 accidents in a year = P (X > 3)

= 1 – P (X ≤ 3)

= 1 – [P (X = 0) + P (X = 1) + P (X = 2) + (X = 3)]

$= 1 - \left[\frac{e^{-3}\times 3^0}{0!} + \frac{e^{-3}\times 3^1}{1!} + \frac{e^{-3}\times 3^2}{2!} + \frac{e^{-3}\times 3^3}{3!}\right]$

$= 1 - e^{-3}\left[\frac{1}{1} + \frac{3}{1} + \frac{9}{2} + \frac{27}{6}\right]$

= 0.3528

Number of drivers with more than 3 accidents in a year = N × P (X > 3)

= N × P (X> 3)

= 352.6

≈ 353 drivers

Hence, in a year

Number of drivers with no accidents = 50

Number of drivers with more than 3 accidents = 353

3 b) Calculate Rank Correlation co-efficient for the following data: **(07)**

X	10	12	18	18	15	40
Y	12	18	25	25	50	25

Ans: Spearman's rank correlation Co-efficient :

X	Y	R_1	R_2	$d_i = (R_1 - R_2)^2$
10	12	6	6	0
12	18	5	5	0
18	25	2.5	3	0.25

18	25	2.5	3	0.25
15	50	4	1	9
40	25	1	3	4
			Total	13.5

Here, m_1 = 2; m_2 = 3; n = 6

Spearman's rank correlation Co-efficient is given by

$\boldsymbol{R} = 1 - \frac{6}{n(n^2-1)}\left\{\sum d_i^2 + \frac{1}{12}\left[\left(m_1^3 - m_1\right) + \left(m_2^3 - m_2\right) + \cdots.\right]\right\}$

$= 1 - \frac{6}{6(6^2-1)}\left\{13.5 + \frac{1}{12}\left[(2^3 - 2) + (3^3 - 3)\right]\right\}$

$= 1 - \frac{1}{35}\left\{13.5 + \frac{1}{12}[6 + 24]\right\}$

$= 1 - \frac{1}{35}\{13.5 + 2.5\}$

= 0.5429

Spearman's rank correlation Co-efficient (R) = 0.5429

3 c) Expand $f(z) = \frac{1}{z^2(z-1)(z+2)}$ about z = 0 for **(08)**

i) $|z| < 1$;

ii) $1 < |z| < 2$;

iii) $|z| > 2$.

Ans: Let $f(z) = \frac{1}{z^2(z-1)(z+2)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} + \frac{D}{z+2} \rightarrow$ (1)

By cover-up method, $B = \frac{-1}{2}$; $C = \frac{1}{3}$; $D = \frac{-1}{12}$;

Put z= -1 in (1)

$\frac{1}{(-1)^2(-1-1)(-1+2)}$ = $\frac{A}{-1} - \frac{1/2}{(-1)^2} + \frac{1/3}{-1-1} - \frac{1/12}{-1+2}$

$\therefore -\frac{1}{2} = -A - \frac{1}{2} - \frac{1}{6} - \frac{1}{12}$

$$\therefore A = -\frac{1}{4}$$

$$\therefore f(z) = \frac{-1}{4z} - \frac{1}{2z^2} + \frac{1}{3(z-1)} - \frac{1}{12(z+2)} \rightarrow (2)$$

Case.1: For $|z| < 1$,
Obviously, $|z| < 2$

$\therefore |z| < 1$ and $\left|\frac{z}{2}\right| < 1$

$\therefore$ From (2)

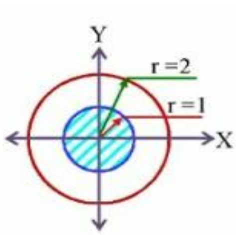
$f(z) = \frac{-1}{4z} - \frac{1}{2z^2} + \frac{1}{3(z-1)} - \frac{1}{12\times 2\,(z/2+1)}$

$= \frac{-1}{4z} - \frac{1}{2z^2} - \frac{1}{3}(1-z)^{-1} - \frac{1}{24}\left(1+\frac{z}{2}\right)^{-1}$

$= \frac{-1}{4z} - \frac{1}{2z^2} - \frac{1}{3}(1+z+z^2+z^3+\cdots) - \frac{1}{24}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^2}{2^3}+\cdots\right)$

Region of Convergence:

Above series is convergent for
$|z| < 1$ and $|z| < 2$ i.e. $|z| < 1$,



Which is the interior of the circle with centre (0,0) and radius 1,

Case 2: For $1 < |z| < 2$

Obviously, $|z| > 1$

i.e. $1 < |z|$ and $|z| < 2$

$\therefore \left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$

$\therefore$ From (2)

$$f(z) = \frac{-1}{4z} - \frac{1}{2z^2} - \frac{1}{3z(1-1/z)} - \frac{1}{12\times2(z/2+1)}$$

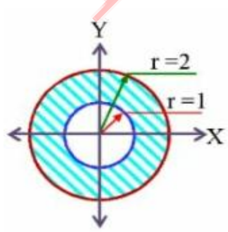
$$= \frac{-1}{4z} - \frac{1}{2z^2} + \frac{1}{3z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{24}\left(1+\frac{z}{2}\right)^{-1}$$

$$= \frac{-1}{4z} - \frac{1}{2z^2} - \frac{1}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) - \frac{1}{24}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\cdots\right)$$

$$= \frac{-1}{4z} - \frac{1}{2z^2} - \frac{1}{3}\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \frac{1}{24}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\cdots\right)$$

Region of Convergence:

Above series is convergent for



$\left|\frac{1}{z}\right| < 1$ & $\left|\frac{z}{2}\right| < 1$

i.e. 1 <|z| < 2, which is the annular region between the concentric circles with Centre (0,0) and radii 1 & 2.

Case 3: For |z|> 2.

Obviously, |z| > 1

i.e. 1 <|z| and 2 <|z|

$\therefore \left|\frac{1}{z}\right| < 1$ and $\left|\frac{2}{z}\right| < 1$

∴ From (2)

$$f(z) = \frac{-1}{4z} - \frac{1}{2z^2} + \frac{1}{3z(1-1/z)} - \frac{1}{12z(1+2/z)}$$

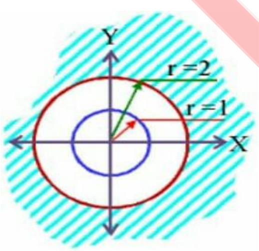
$$= \frac{-1}{4z} - \frac{1}{2z^2} + \frac{1}{3z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{12z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= \frac{-1}{4z} - \frac{1}{2z^2} - \frac{1}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) - \frac{1}{12z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\cdots\right)$$

$$= \frac{-1}{4z} - \frac{1}{2z^2} - \frac{1}{3}\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \frac{1}{12}\left(\frac{1}{z}-\frac{2}{z^2}+\frac{2^2}{z^3}-\cdots\right)$$

Region of Convergence:

Above series is convergent



For $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{2}{z}\right| < 1$

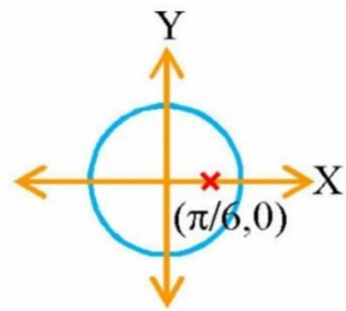
i.e |z| > 2, which is the exterior region of the circle with centre (0,0) and radius 3.

4.a) Evaluate $\int_c \frac{sin^{\theta}z}{(z-\pi/6)^n}\, dz$ where c is circle |z| = 1 for n= 1, n = 3 (06)

Ans: Circle |z| =1 has Centre (0,0) and radius 1.

Here, $z_0 = \frac{\pi}{6}$ lies inside the circle .

Case.I. n = 1



$z_0 = \frac{\pi}{6}$ is a simple pole.

R_1 = Residue of f(z) at " $z_0 = \frac{\pi}{6}$"

$= \lim_{z\to z_0}(z - z_0)f(z)$

$= \lim_{z\to \pi/6}(z - \pi/6) \times \frac{sin^6 z}{(z-\frac{\pi}{6})}$

$= sin^6 \frac{\pi}{6}$

$= \frac{1}{64}$

By Cauchy's Residue theorem,

$$\int_c f(z)dz = 2\pi i(R^1 + R^2 + \cdots)$$

$$\therefore \int_c \frac{sin^6 z}{(z-\pi/6)^1}\, dz = 2\pi i.\frac{1}{64}$$

$$\therefore \int_c \frac{\boldsymbol{sin^6 z}}{\boldsymbol{(z-\pi/6)^1}}\, \boldsymbol{dz} = \frac{\boldsymbol{\pi i}}{\boldsymbol{32}}$$

Case II: n =3

$z_0 = \frac{\pi}{6}$ is a pole of order 3

R_1 = Residue of $f(z)$ at "$z_0 = \frac{\pi}{6}$"

$= \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n \times f(z)$

$= \frac{1}{(3-1)!} \lim_{z \to \pi/6} \frac{d^{3-1}}{dz^{3-1}} (z - \pi/6)^3 \times \frac{sin^6 z}{(z-\pi/6)^3}$

$= \frac{1}{2!} \lim_{z \to \pi/6} \frac{d^2}{dz^2} sin^6 z$

$= \frac{1}{2} \lim_{z \to \pi/6} \frac{d}{dz}(6\, sin^5\, z . \cos z)$

$= \frac{1}{2} \times 6 \lim_{z \to \pi/6} (\sin^5 z . - \sin z + \cos z\ .5 \sin^4 z\ .\cos z)$

$= 3 \left(-sin^6 \frac{\pi}{6} + cos^2 \frac{\pi}{6}\ .5\ sin^2 \frac{\pi}{6}\right)$

$= 3 \left(-\frac{1}{64} + \frac{3}{4}\ .5 . \frac{1}{16}\right)$

$= \frac{21}{32}$

By Cauchy's Residue theorem .

$\int_c f(z)dz = 2\pi i(R_1 + R_2 + \cdots)$ 1

$$\therefore \int_c \frac{sin^6 z}{(z - \pi/6)^3}\, dz = 2\pi i\ . \frac{21}{32}$$

$$\therefore \int_c \frac{\boldsymbol{sin^6 z}}{(\boldsymbol{z - \pi/6})^3}\, \boldsymbol{dz} = \frac{\boldsymbol{21\pi i}}{\boldsymbol{16}}$$

4 b) Find the m.g.f of a random variable whose probability density function is **(06)**

$$P\ (X\ =\ x) = \begin{cases} \left(\frac{1}{2}\right)^x & x = 1, 2, 3, \ldots \\ 0 & elsewhere \end{cases}$$

Hence ,find the mean variance.

Ans: By definition M.G.F. about origin is given by

$$M_o(t) = E[e^{tx}]$$

= $\sum_{x=1}^{\infty}\ P_x\,.e^{tx}$

= $\sum_{x=1}^{\infty}\ \frac{1}{2^x}.e^{tx}$

= $\sum_{x=1}^{\infty}\ \left(\frac{e^t}{2}\right)^x$

= $\left(\frac{e^t}{2}\right)^1 + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots,$

which is a Geometric Progression with a= $\frac{e^t}{2}$and r = $\frac{e^t}{2}$

= $\frac{e^t/2}{1-e^t/2}$, $\left\{In\ G, P., s_\infty = \frac{a}{1-r}\right\}$

= $\frac{e^t}{2-e^t}$

=$\frac{e^t}{e^t\left(\frac{2}{e^t}-1\right)}$

$$\therefore M_o(t) = \frac{1}{2e^{-t} - 1}$$

Now, r^{th} Moments $\mu_r{}'$ = $\left[\frac{d^r}{dt^r}M_o(t)\right]_{t=0} \rightarrow (1)$

$\therefore$ First Moment $\mu_r{}'$ = $\left[\frac{d^1}{dt^1}(2e^{-t} - 1)^{-1}\right]_{t=0}$

$= \left[-1(2e^{-t}-1)^{-2}.\frac{d}{dt}(2e^{-t}-1)\right]_{t=0}$

$= [-1(2e^{t}-1)^{-2}.2e^{t}.-1]_{t=0} \rightarrow (2)$

$= (2e^{0}-1)^{-2}.\,2e^{0}$

$= 1^{-2} \times 2 \times 1$
= 2

From (1),Second Moment $\mu_2{}' = \left[\frac{d^2}{dt^2}M_o(t)\right]_{t=0}$
$= \left\{\frac{d}{dt}.[2e^{-t}(2e^{-t}-1)^{-2}]\right\}_{t=0}$ (From 2)

$= 2\left[e^{-t}\frac{d}{dt}(2e^{-t}-1)^{-2} + (2e^{-t}-1)^{-2}\frac{d}{dt}e^{-t}\right]_{t=0}$

$= 2[e^{-t}.-2(2e^{-t}-1)^{-3}.2e^{-t}.-1 + (2e^{-t}-1)^{-2}.e^{-t}.-1]_{t=0}$

$= 2\,[2e^{0}(2e^{0}-1)^{-3}.2e^{0} - e^{0}(2e^{0}-1)^{-2}]$

$= 2\,[2 \times 1^{-3}.2 \times 1 - 1 \times 1^{-2}]$

= 2[4-1]

= 6
∴ Mean = μ_1' = 2

∴ Variance = $\mu_2' - \mu_1'^2$

$= 6 - 2^2$
= 2
Hence
Mean = 2 and Variance = 2

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4 c) Verify the Cayley-Hamilton Theorem for matrix A and hence find A^{-1} for **(08)**
A = $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.
Hence find $A^5 - 2A^4 - 7A^3 + 11A^2 - A - 10I$ interms of A.

Ans: **Part I:**

Let λ be eigen value of matrix A.

Characteristic equation is $|A - \lambda I| = 0$

$\therefore \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$

On solving we get

λ^2 - (sum of diagonal elements) $\lambda - |A| = 0$

$\therefore \lambda^2 + (1+3)\lambda + (-5) = 0$

$$\therefore \lambda^2 + 4\lambda - 5 = 0 \rightarrow (1)$$

Cayley Hamilton Theorem states that the characteristic equation is satisfied by matrix A.

$$\therefore A^2 + 4A - 5I = 0 \rightarrow (2)$$

Now, $A^2 = A \times A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$

$\therefore$ LHS = $A^2 + 4A - 5I$

= $\begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

= $\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$

= RHS

$\therefore$ Cayley Hamilton Theorem is verified.

Part II:

Pre-multiply (2) by A^{-1}

$$\therefore A - 4I - 5A^{-1} = 0$$

$\therefore 5A^{-1} = A - 4\,I$

$\therefore 5A^{-1} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore 5A^{-1} = \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$

$$\therefore \boldsymbol{A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}}$$

Part III.

Consider,

$$\begin{array}{r|l} & \lambda^3 - 2\lambda + 3 \\ \hline \lambda^2 - 4\lambda - 5 & \lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 \\ & \lambda^5 - 4\lambda^4 - 5\lambda^3 \\ \hline & -2\lambda^3 + 11\lambda^2 - \lambda \\ & -2\lambda^3 + 8\lambda^2 + 10\lambda \\ \hline & 3\lambda^2 - 11\lambda - 10 \\ & 3\lambda^2 - 12\lambda - 15 \\ \hline & \lambda + 5 \end{array}$$

Divided = Divisor $\times$ Quotient + Remainder

$$\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 = (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3)(\lambda + 5)$$

= 0 + $(\lambda + 5)$ (from 1)
=$\lambda + 5$

$$\therefore \quad \boldsymbol{A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5I}$$

5 a) Express $p(x)$ = 7 + $8x$ + $9x^2$ as a Liner combination of p_1 = 2 + x + $4x^2$; p_2 =1- x + $3x^2$ p_3 = 2 + x + $5x^2$.
(06)

Ans: $p(x)$ = 7 + $8x$+ $9x^2$= (7,8,9)
p_1 = 2 + x + 4 x^2 = (2,1,4)
p_2 = 1 + x + 3 x^2 = (1,-1,3)
p_3 = 2 + x + 5 x^2 = (2,1,5)

Let k_1, k_2, k_3 be scalers (not all zero)such that

$p(x) = k_1p_1(x) + k_2p_2(x) + k_3p_3(x) \rightarrow$ (1)

∴ (7,8,9) = k_1(2,1,4) + k_2 (1,-1,3) + k_3 (2,1,5)

∴ (7,8,9) =

$(2k_1, 1k_1, 4k_1)$+ $(1k_2, -1k_2, 3k_2)$+$(2k_3, 1k_3, 5k_3)$

∴ (7,8,9) =

($2k_1 + 1k_2 + 2k_3$, $1k_1 - 1k_2$ + $1k_3$.$4k_1$ +$3k_2$ + $5k_3$)

∴ Comparing both sides, we get

7 = 2 $k_1 + 1k_2 + 2k_3$
8 = 1 $k_1 - 1k_2 + 1k_3$
9 = 4 $k_1 + 3k_2 + 5k_3 \rightarrow$ (2)

Writing equation in (2) in the matrix form ,

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & -1 & 1 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

$$R_2 \leftrightarrow R_1 \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 9 \end{bmatrix}$$

$R_2 - 2R_1; R_3 - 4R_1; \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 7 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ -23 \end{bmatrix}$

$\frac{1}{3} R_2; \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ -23 \end{bmatrix}$

R_3 - $7R_2$: $R_1 + R_2; \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ -2 \end{bmatrix}$

On expansion ,

$k_3 = -2$, $k_2 = -3$ and ,
$1k_1 + 0k_2 + 1k_3; = 5$
$\therefore k_1 + 0 - 2 = 5$
$\therefore k_1 = 7$

$\therefore$ From (1) , $p\ (x) = 7\ p_1(x) - 3p_2(x) - 2p_3(x)$
Which is the required Linear combination.

5 b) Using Cauchy theorem ,evaluate $\int_0^{2\pi} \frac{cos2\theta}{5 + 4cos\theta}$ (06)

Ans:Consider a circle |z| = 1 with Centre (0,0) & radius 1.
Put $z = r\, e^{i\theta} = 1e^{i\theta} = e^{i\theta}$
$\therefore dz = e^{i\theta}.id\theta = izd\theta$
$\therefore d\theta = \frac{dz}{iz}$

In Denominator ,cosθ = $\frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} = \frac{z^2+1}{2z}$

In Numerator,
cos 2θ = R.P ($e^{i2\theta}$) = R.P $(e^{i\theta})^2$ = R.P (z^2)

On substituting we get,

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4cos\theta}\, d\theta = \int_c \frac{R.P.z^2}{5\times 4\times\frac{(z^2+1)}{2z}} \cdot \frac{dz}{iz}$$

$$= \text{R.P} \int_c \frac{z\,z^2}{5z+2(z^2+1)} \cdot \frac{dz}{iz}$$

$$= \text{R.P} \int_c \frac{z^2}{2z^2+5z+2} \cdot \frac{dz}{i}$$

Now for singularity, $2z^2 + 5z + 2$

$$\therefore\ z = -0.5 \text{ or } z = -2$$

Here, z_0 = -2 Lies outside while z_0 = -0.5 Lies inside the circle |z|=1

$z_0 = -0.5$ is a simple pole.

R_1 = Residue of $f(z)$ at "$z_0 = -0.5$"

$= \lim_{z\to z_0} (z - z_0) \times f(z)$

$$= \lim_{z\to 0.5} (z + 0.5) \times \frac{z^2}{i\times 2(z+0.5)(z+2)}$$

$$= \frac{(-0.5)^2}{i\times 2(-0.5+2)}$$

$$= \frac{-i}{12}$$

By Cauchy 's Residue theorem ,

$$\int_c f(z)dz = 2\pi i(R_1 + R_2 + \cdots)$$

$$\therefore \text{R.P.} \int_c \frac{z^2}{2z^2+5z+2} \cdot \frac{dz}{i} = R.P\left[2\pi i.\left(\frac{-i}{12}\right)\right]$$

$$\therefore \int_0^{2\pi} \frac{cos2\theta}{5+4cos\theta}\, d\theta = R.P.\left(\frac{\pi}{6}\right)$$

$$\therefore \int_0^{2\pi} \frac{\boldsymbol{cos2\theta}}{\boldsymbol{5 + 4cos\theta}}\, \boldsymbol{d\theta} = \frac{\boldsymbol{\pi}}{\boldsymbol{6}}$$

5 c) In an examination marks obtained by students in Mathematics ,physics and Chemistry are normally distributed with means 51, 53 and 46 respectively and standard deviation 15,12,16 respectively .Find the probability of securing total marks. (08)

i) 180 or above

ii) 90 or below

Ans:

Let M,P and C denote marks in Mathematics, Physics and chemistry respectively

Given $m_M = 51; m_P = 53; m_C = 46; \sigma_M = 15; \sigma_P = 12; \sigma_C = 16$

Let X = Total Marks of Mathematics ,Physics and Chemistry

∴ X = M + P + C

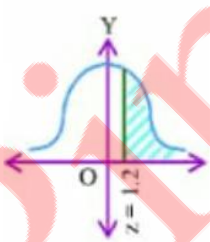
Since,M,P,C are independent normal variates ,X is also a normal variate with,

Mean = $m_M + m_P + m_C = 51 + 53 + 46 = 150$

Variance (σX^2) = $\sigma M^2 + \sigma P^2 + \sigma C^2$ = $15^2 + 12^2 + 16^2$= 625

∴ σ_X = 25

i)P (180 or above)= P (X> 180)



= $P\left(\frac{x-m}{\sigma} > \frac{180-150}{25}\right)$

=P(z> 1.2)

= 0.5 – Area between ‘z=0’ to ‘z= 1.2’

= 0.5 – 0.3849

= 0.1151

ii) P (90 or below)= P (X< 90)

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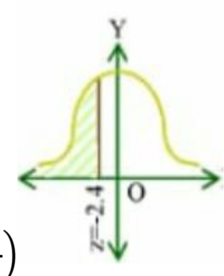


$= P\left(\frac{x-m}{\sigma} < \frac{90-150}{25}\right)$

$= P\ (z < -2.4)$
= 0.5 – Area between 'z=0' to 'z= - 2.4'
= 0.5 – 0.4918
= 0.0082

Hence,

- **i) Probability of securing total marks of 180 and above = 0.1151**
- **ii) Probability of securing total marks of 90 and below = 0.0082**

6 a) If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then find A^{50}.

Ans: Let λ be eigen value of matrix A .
Characteristic equation is $|A - \lambda I| = 0$

$\therefore \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$

On solving we get
$\lambda^3 -$ (Sum of diagonal elements)$\lambda^2 +$ (sum of the minors of diagonal elements) $\lambda - |A| = 0$

$$\therefore \lambda^3 - (1+0+0)\lambda^2 + (1+0+0)\lambda - (-1) = 0$$

$$\therefore \lambda^3 - \lambda^2 + \lambda + 1 = 0$$

$\therefore$ Eigen values (λ) are 1,1,-1

Since,A is order 3× 3, let

$f(A) = A^{50} = a\,A^2 + bA \; + \; cI \rightarrow (1)$ (where a, b,c are constants)

We assume f (A) is satisfied by λ

$$\therefore \lambda^{50} = a\lambda^2 + b\lambda + c \rightarrow (2)$$

Put $\lambda = 1$,

$$(-1)^{50} = a(-1)^2 + b(-1) + c$$

$\therefore$ 1 = a – b + c → (3)

Put $\lambda = 1\; in\; (2)$

$$1^{50} = a(1)^2 + b(1) + c$$

$\therefore$ 1 = a + b + c → (4)

Differentiating (2) w. r.t λ, $50\,\lambda^{49} = 2a\lambda + b$

When $\lambda = -1\,, 50\;(-1)^{49} = 2a(-1) + b$

$\therefore$ -50 = - 2a + b→ (5)

Solving (3), (4) and (5) simultaneously , we get
a = 25,b = 0, c = -24

Substituting in (I) ,we get ,

$$A^{50} = 25A^2 + 0\,A - 24\,I$$

$$\therefore A^{50} = 25A \times A - 24I$$

$= 25\begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} - 24\,I$

$= 25\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\end{pmatrix} - 24\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

$=\begin{pmatrix}1 & 0 & 0\\ 25 & 1 & 0\\ 25 & 0 & 1\end{pmatrix}$

Hence, $A^{50} = \begin{pmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{pmatrix}$

6 b) Using residue theorem evaluate $\int_C \frac{e^z}{(z^2+\pi^2)^2}\, dz$, *where C is* $|z| = 4$ **(06)**

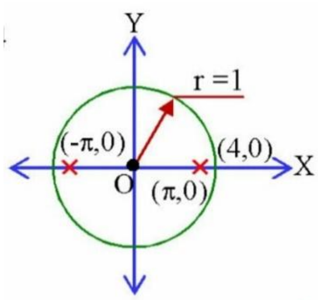
Ans. The circle |z| = 4 has centre (0,0) & radius 4

For singularity, $z^2 + \pi^2 = 0$

$\therefore z^2 = -\pi^2 = i^2\pi^2$

$\therefore z = \pm\, i\, \pi$

Here, "$z_0 = \pm\, i\, \pi$" lies inside the circle



z_0 = "$\pm$ i π" are poles of order 2

R_1 = Residue of f (z) at "$z_0 = i\,\pi$"

$$= \frac{1}{(n-1)!} \lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n \times f(z)$$

$= \frac{1}{(2-1)!} \lim_{z\to i\pi} \frac{d}{dz} (z - i\pi)^2 \times \frac{e^z}{(z-i\pi)^2(z+i\pi)^2}$

$= \frac{1}{1!} \lim_{z\to i\pi} \frac{d}{dz} e^z (z + i\pi)^{-2}$

$= \lim_{z\to i\pi} e^z . -2(z + i\pi)^{-3} + (z + i\pi)^{-2} . e^z$

$= \frac{-2e^{i\pi}}{(i\pi+i\pi)^3} + \frac{e^{i\pi}}{(i\pi+i\pi)^3}$

Now, $e^{\pm i\pi}$ = cosπ $\pm i$ sinπ = $-1 \pm 0 = -1$

$$\therefore R_1 = \frac{2}{8i^3\pi^3} - \frac{1}{4i^2\pi^2}$$

Similarly, R_2 = Residue of f (z) at " $z_0 = -i\pi$"

= $\frac{-2e^{-i\pi}}{(-i\pi - i\pi)^3} + \frac{e^{-i\pi}}{(-i\pi - i\pi)^2}$

$$\therefore R_2 = \frac{2}{8i^3\pi^3} - \frac{1}{4i^2\pi^2}$$

By Cauchy's Residue theorem, $\int_c f(z)dz = 2\pi i(R_1 + R_2 + \ ...)$

$\therefore \int_c \frac{e^z}{(z^2+\pi^2)^2}\, dz = 2\pi i\left(\frac{2}{8i^3\pi^3} - \frac{1}{4i^2\pi^2} - \frac{2}{8i^3\pi^3} - \frac{1}{4i^2\pi^2}\right)$

$$= 2\pi i \times \frac{-2}{4i^2\pi^2}$$

$\therefore \int_c \frac{e^z}{(z^2+\pi^2)^2}\, dz = \frac{i}{\pi}$

6 c) Using Rayleigh- Ritz method solve the boundary value problem

$I = \int_0^1 \left[xy + \frac{1}{2}y'^2\right] dx;\ 0 \le x \le 1$,given y (0)= y (1)= 0 where $\bar{y}\,(x) = c_0 + c_1 x + c_2 x^2$.

Ans:$I = \int_0^1 \left[xy + \frac{1}{2}y'^2\right] dx \rightarrow (1)$

Let the approximate solution be
$y\,(x) = C_0 + C_1 x + C_2 x^2 \rightarrow (2)$

Putting x =0, y (0)= $C_0 + 0 + 0$
$\therefore$ 0 = $C_0 \rightarrow (3)[\because y(0) = 0]$

Putting x = 1 in (2), y (1) = $C_0 + C_1 + C_2 [\because y(1) = 0]$

$\therefore$ 0 = 0 + $C_1 + C_2 [From\ 3]$

$\therefore C_2 = -C_1 \rightarrow (4)$

Substituting (3) and (4) in (2),

$y = 0 + C_1 x - C_1 x^2 \rightarrow (5)$

Differentiating w.r.t. 'x', $y' = c_1 - 2c_1 x \rightarrow (6)$

Substituting (5) and (6) in (1), we get

$I = \int_0^1 \left[x\left([c_1 x - c_1 x^2] + \frac{1}{2}(c_1 - 2c_1 x)^2\right)\right] dx$

$= \int_0^1 \left[c_1 x^2 - c_1 x^3 + \frac{1}{2}({c_1}^2 - 4{c_1}^2 x + 4{c_1}^2 x^2)\right] dx$

$= \int_0^1 \left[c_1 x^2 - c_1 x^3 + \frac{1}{2}{c_1}^2 - 2{c_1}^2 x + 2{c_1}^2 x^2\right] dx$

$= \left[\frac{c_1 x^3}{3} - \frac{c^1 x^4}{4} + \frac{1}{2}{c_1}^2 x - \frac{2{c_1}^2 x^2}{2} + \frac{2{c_1}^2 x^3}{3}\right]_0^1$

$= \left[\frac{c_1}{3} - \frac{c_1}{4} + \frac{1}{2}{c_1}^2 - {c_1}^2 + \frac{2}{3}{c_1}^2\right] - [0 - 0 + 0 - 0 + 0]$

$= I = \frac{c_1}{12} + \frac{1}{6}{c_1}^2$

For maximum or minimum, $\frac{dI}{dc_1} = 0$

$\therefore \frac{dI}{dc_1} = \frac{1}{12} + \frac{1}{6} \times 2c_1 = 0$

$\therefore \frac{1}{3} c_1 = \frac{-1}{12}$

$\therefore c_1 = \frac{-1}{4}$

$\therefore$ From (5), $y = \frac{-1}{4} x + \frac{1}{4} x^2$

Hence, the approximate solution is $y = \frac{x}{4}(x - 1)$